The shell theorem in non metric physical relativistic gravitation theory

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Two criteria may be used to define the shell theorem for a gravitational field model; firstly that the external field of a thin uniform spherical shell of mass be the same as if its mass were concentrated at its center, and secondly that the field interior to the shell be zero. Other authors have shown that only a limited set of gravitational potentials satisfy one or the other of these two criteria and that only the 1/r Newtonian potential satisfies them both in 3 spatial dimensions. It is shown here however that the exponential potential law from a recently developed relativistic gravitational theory also satisfies both criteria of the shell theorem.

I. INTRODUCTION

The shell theorem addresses the exterior and interior gravitational fields of a uniform spherical thin shell of mass. The result of the theorem for a given gravitational model is extremely useful for describing the field of a distribution of mass.

It is well known in Newtonian gravitation that the external gravitational field of a thin uniform spherical shell of mass M is equivalent to that of a point mass M located at its center. Furthermore Newton originally showed[3] the gravitational field interior to the shell is zero. These together have become known as Newton's shell theorem.

Subsequently, authors treating the converse of that theorem[1] have concluded that only certain central force laws satisfy either or both of the two conditions - that the field exterior to the shell is that of a point mass and that the interior field is zero. In this reference potential functions of the form $g(r)/r$ were assumed where $g(r)$ is a general functional replacement for Newton's gravitational constant. It concludes that the only force law satisfying both of these conditions in 3 spatial dimensions is the inverse square law.

While this conclusion may be accurate as far as it goes, it is restricted to the class of potential laws that were assumed. A recent publication[2] describes a relativistic theory of gravitation that reduces to general relativity and Newtonian gravitation in weak field limits. It will be referred to here as exponential relativity or exponential gravitation theory. Within this theory the gravitational energy U of a test mass m_0 in the gravitational field of mass M is given by the exponential function in spherical coordinates

$$U = m_0c^2 \left(e^{-\frac{r_g}{r}} - 1\right) \tag{1}$$

where $r_g = GM/c^2$, and r is the radial coordinate of the test mass m_0 in a spherical coordinate system centered on mass M. The universal gravitational constant is given by G, and c is the speed of light. The purpose of this article is to show this exponential potential satisfies both criteria of the shell theorem even though it is not an inverse square field law.

II. THE PROOF

1. Physical Relativity Gravitation Summary

A brief summary of the derivation of the potential energy function from physical relativity is provided here. It will also be referred to as the exponential potential energy function. If a free test mass at rest has a self energy of m_0c^2 and has total energy E of $m_0c^2 + U$ when in the field of mass M, then we may associate a mass m of this test particle according to Einstein's mass-energy equivalence, or

$$m = \frac{m_0c^2 + U}{c^2} = m_0 + \frac{U}{c^2} \tag{2}$$

We may write a differential of potential energy when the particle is in a gravitational force field satisfying Newton's gravitational law as

$$dU = \frac{GMm}{r^2} dr \tag{3}$$

Substituting equation (2) into the equation above gives for a unit test mass m_0

$$dU = GM(1 + \frac{U}{c^2})\frac{dr}{r^2} \tag{4}$$

$$\frac{dU}{(1 + \frac{U}{c^2})} = \frac{c^2 \frac{1}{c^2} dU}{(1 + \frac{U}{c^2})} = GM\frac{dr}{r^2} \tag{5}$$

$$\frac{\frac{1}{c^2} dU}{(1 + \frac{U}{c^2})} = \frac{GM}{c^2}\frac{dr}{r^2} \tag{6}$$

Substituting $r_g = GM/c^2$, and using the initial condition that $U = 0$ at $r = \infty$ the solution is

$$U = c^2(e^{-\frac{r_g}{r}} - 1) \tag{7}$$

And for a non-zero test mass m_0

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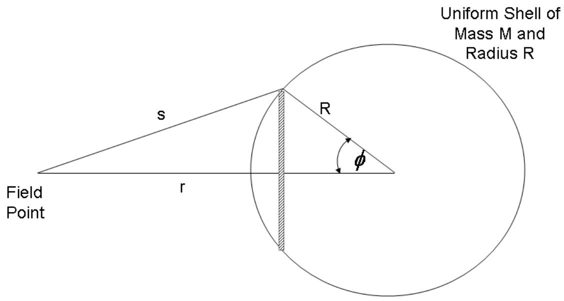

FIG. 1. Exterior Field of Shell of Mass M

$$U = m_0 c^2 (e^{-\frac{r_g}{r}} - 1) \tag{8}$$

Reference[2] extends this result to develop a relativistic theory of gravitation. We use it here however to describe gravitational potential and note that by taking the first two terms of a Taylor expansion of the exponential function we have

$$U \approx m_0 c^2 (1 - \frac{r_g}{r} + ... - 1) \tag{9}$$

$$U = -\frac{GMm_0}{r} \tag{10}$$

showing that the exponential potential of equation (8) reduces to Newtonian potential energy in the weak field limit.

2. Application of exponential potential energy function to the shell theorem

Consider a test mass in the potential energy field of a uniform spherical shell of mass M as shown in figure 1. Assigning an aerial mass density σ to the shell of $M/(4\pi R^2)$ we may then write the differential of potential energy from the exponential potential energy function of equation (8) for the indicated ring of mass as

$$dU = m_0 c^2 e^{-\frac{r_g}{s}} \left(-\frac{dr_g}{s}\right) \tag{11}$$

$$dU = (U + m_0 c^2)\left(-\frac{d(2\pi G\sigma R \sin\phi R d\phi)}{c^2 s}\right) \tag{12}$$

$$dU = (U + m_0 c^2)\left(\frac{GM}{2c^2}\right)\frac{d\left(\cos(\phi)\right)}{s} \tag{13}$$

From the law of cosines

$$s^2 = R^2 + r^2 - 2rRcos(\phi) \tag{14}$$

$$cos(\phi) = -\frac{s^2 - R^2 - r^2}{2rR} \tag{15}$$

$$d(cos(\phi)) = -\frac{s}{rR}ds \tag{16}$$

Substitution of this into equation (13) and simplifying

$$\frac{dU}{U + m_0 c^2} = -\left(\frac{GM}{2c^2}\right)\frac{ds}{rR} \tag{17}$$

3. Exterior Field

For a test point exterior to the shell, we integrate between the indicated definite limits and obtain

$$\int_0^U \frac{dU}{U + m_0 c^2} = -\frac{GM}{2c^2 rR}\int_{r-R}^{r+R} ds \tag{18}$$

$$\ln\left(\frac{U + m_0 c^2}{m_0 c^2}\right) = -\frac{GM}{2c^2 rR}(r + R - r + R) \tag{19}$$

$$\ln\left(\frac{U}{m_0 c^2} + 1\right) = -\frac{GM}{c^2 r} \tag{20}$$

Or

$$U = m_0 c^2 \left(e^{-\frac{GM}{c^2 r}} - 1\right) \tag{21}$$

$$U = m_0 c^2 \left(e^{-\frac{r_s}{r}} - 1\right) \tag{22}$$

This is the same potential as if all the shell's mass were concentrated into a point at distance r from the test mass as given in equation (8).

The negative spatial gradient of this potential, the field strength, is therefore the field law of a single point mass thereby proving the first condition of the shell theorem is met.

4. Interior Field

For a test point interior to the shell, equation (18) is still valid, but the integral limits change to

$$\int_0^U \frac{dU}{U + m_0 c^2} = -\frac{GM}{2c^2 rR}\int_{R-r}^{R+r} ds \tag{23}$$

And as before

$$\ln(U + m_0 c^2) - \ln(m_0 c^2) = -\frac{GM}{2c^2 rR}(R + r - R + r) \tag{24}$$

$$\ln\left(\frac{U}{m_0 c^2} + 1\right) = -\frac{GM}{2c^2 rR}(2r) = -\frac{GM}{c^2 R} \tag{25}$$

Or

$$U = m_0 c^2 \left(e^{-\frac{r_s}{R}} - 1 \right) \tag{26}$$

It may be seen the potential inside the spherical shell does not depend on the coordinate r. Hence the field, the spatial gradient of U, is zero. This proves the second criterion of the shell theorem.

III. CONCLUSIONS

The exponential gravitation potential function satisfies both conditions of the shell theorem, namely that the external gravitational field of a thin uniform shell of mass M is identical to that of a mass point M located at the center of the shell, and the interior field is zero.

[1] **Arens, Richard** "Newton's observations about the field of a uniform thin spherical shell" *Note di Matematica X (Suppl. n. 1): 3945*, 1990

[2] **Douglass, James W.** "A nonmetric theory of gravitation that is nonsingular at the Schwarzschild radius" *arXiv:1508.05810 [gr-qc]*, 2015

[3] **Newton, Isaac**, *Philosophiae Naturalis Principia Mathematica* London, 1687